It is important to remember that all of these results, including Wick's theorem, are true *only* for quantum Gaussian density operators, and that for most interacting systems such density operators are normally not valid. To use these results more generally one must make approximations.

10.5 Coherent States

The study of both coherent light and of ultra-cold Bose gases is necessarily the study of highly occupied states of Bose particles. The two most important kinds of such states are *number states* and *coherent states*. Number states are eigenstates of the number operator $N = a^\dagger a$. Since in quantum-optical situations, photons are not conserved, a photon number state is very hard to create and preserve, whereas a coherent state is much more robust.

In an ultra-cold gas, the fact that the total number of atoms in a gas is absolutely conserved might lead one think that number states would be the most relevant for their study. Surprisingly, this is not so, and the coherent state $|\alpha\rangle$, which satisfies the defining eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle, \qquad (10.5.1)$$

and does not correspond to a definite number of particles, still provides a valuable tool for the study of ultra-cold gases.

10.5.1 Properties of the Coherent States

a) Expression in Terms of Number States: The only solution to the defining equation (10.5.1), which also satisfies $\langle\alpha|\alpha\rangle = 1$ is (up to a phase)

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \qquad (10.5.2)$$

b) Action of Creation and Destruction Operators: Using $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, it is straightforward to show that

$$\left.\begin{aligned} a|\alpha\rangle &= \alpha|\alpha\rangle, \\ a^\dagger|\alpha\rangle &= \left(\alpha^* + \tfrac{1}{2}\frac{\partial}{\partial\alpha}\right)|\alpha\rangle, \\ \langle\alpha|a^\dagger &= \langle\alpha|\alpha^*, \\ \langle\alpha|a &= \left(\alpha + \tfrac{1}{2}\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|. \end{aligned}\right\} \qquad (10.5.3)$$

c) Unitary Transformation of the Vacuum: We can also show that

$$|\alpha\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle. \tag{10.5.4}$$

This involves the use of the Baker–Hausdorff formula: For any two operators A and B, such that $[A,B]$ commutes with both of them, one can write

$$\exp(A+B) = \exp(A)\exp(B)\exp\left(-\tfrac{1}{2}[A,B]\right), \tag{10.5.5}$$

$$= \exp(B)\exp(A)\exp\left(\tfrac{1}{2}[A,B]\right), \tag{10.5.6}$$

and this is proved in *Quantum Noise.* Using this identity, we see that (10.5.4) gives

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\exp\left(\alpha a^\dagger\right)\exp\left(-\alpha^* a\right)|0\rangle, \tag{10.5.7}$$

and noting that $a|0\rangle = 0$, we see

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\sum \frac{\alpha^n (a^\dagger)^n}{n!}|0\rangle, \tag{10.5.8}$$

which yields (10.5.2) when we use the expression

$$|n\rangle \equiv \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle. \tag{10.5.9}$$

d) Scalar Product:

$$\langle\alpha|\beta\rangle = \exp\left(\alpha^*\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2\right), \tag{10.5.10}$$

$$|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right). \tag{10.5.11}$$

Notice that no two coherent states are actually orthogonal to each other. However, if α and β are significantly different from each other, the two states are almost orthogonal.

e) Completeness Formula:

$$1 = \frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha|. \tag{10.5.12}$$

Here,

$$\alpha = \alpha_x + i\alpha_y, \qquad d^2\alpha = d\alpha_x d\alpha_y, \tag{10.5.13}$$

and the integral is over the whole complex plane.

Exercise 10.2 Trace in Terms of Coherent States: Show that the resolution of the identity (10.5.12) implies that

$$\mathrm{Tr}\{A\} = \frac{1}{\pi}\int d^2\alpha\,\langle\alpha|A|\alpha\rangle. \tag{10.5.14}$$

f) Normal Products: In evaluating matrix elements, *normal products* of operators in which all destruction operators stand to the right of creation operators, are useful. Thus,

$$\langle\alpha|a^\dagger a a^\dagger|\beta\rangle = \langle\alpha|a^\dagger a^\dagger a + a^\dagger[a, a^\dagger]|\beta\rangle, \tag{10.5.15}$$
$$= \langle\alpha|a^\dagger a^\dagger a + a^\dagger|\beta\rangle, \tag{10.5.16}$$
$$= (\alpha^{*2}\beta + \alpha^*)\langle\alpha|\beta\rangle. \tag{10.5.17}$$

The symbol :: around an expression means that it is to be considered a normal product; thus,

$$:(a + a^\dagger)(a + a^\dagger):= a^{\dagger 2} + a^2 + 2a^\dagger a. \tag{10.5.18}$$

From (10.5.1) it follows that the matrix element between coherent states $\langle\alpha|$ and $|\beta\rangle$ of any normally ordered function $F(a^\dagger, a)$ of creation and destruction operators is given by $F(\alpha^*, \beta)$. Thus, for example,

$$\langle\alpha|:(a + a^\dagger)^3:|\beta\rangle = (\beta + \alpha^*)^3. \tag{10.5.19}$$

g) Poissonian Number Distribution of Coherent States: The state $|n\rangle$ is known as an *n-quantum state* and the probability of observing n quanta in a coherent state $|\alpha\rangle$ is

$$P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \left|\exp(-\tfrac{1}{2}\alpha^2)\frac{\alpha^n}{\sqrt{n!}}\right|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}, \tag{10.5.20}$$

which is a Poisson distribution with mean $|\alpha|^2$. Since the number n corresponds to the eigenvalue of the number operator N, we have

$$\langle N\rangle = \langle\alpha|N|\alpha\rangle = \sum_n nP(n) = |\alpha|^2, \tag{10.5.21}$$
$$\langle N^2\rangle = \langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle = \langle\alpha|a^\dagger a^\dagger a a + a^\dagger a|\alpha\rangle = |\alpha|^4 + |\alpha|^2. \tag{10.5.22}$$

Hence,

$$\mathrm{var}[N] = |\alpha|^2 = \langle N\rangle, \tag{10.5.23}$$

as required for a Poisson.

Exercise 10.3 Action of $\exp(\lambda a^\dagger a)$ on a Coherent State: If $\lambda = \gamma + i\nu$, show that the definition (10.5.2) of the coherent state implies that

$$\exp(\lambda a^\dagger a)|\alpha\rangle = \exp\left(\tfrac{1}{2}|\alpha|^2(e^{2\gamma} - 1)\right)\left|\alpha e^\lambda\right\rangle. \tag{10.5.24}$$

10.5.2 The Harmonic Oscillator

The harmonic oscillator Hamiltonian can be written

$$H_{\mathrm{HO}} = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right), \tag{10.5.25}$$

and the result of Ex. 10.3 shows that the coherent state wavefunction

$$|\psi_c, t\rangle \equiv \exp(-\mathrm{i}H_{\mathrm{HO}}t/\hbar)|\alpha_0\rangle, \tag{10.5.26}$$

$$= \mathrm{e}^{-\mathrm{i}\omega t/2}\left|\mathrm{e}^{-\mathrm{i}\omega t}\alpha_0\right\rangle, \tag{10.5.27}$$

is a solution of the equations of motion.

Let us consider the situation when we drive this with an external field. This can be done by the Hamiltonian

$$H_{\mathrm{Driven}} = \hbar\left\{\omega\left(a^\dagger a + \tfrac{1}{2}\right) + g^*(t)a + g(t)a^\dagger\right\}. \tag{10.5.28}$$

The Heisenberg equation of motion for $a(t)$ is

$$\frac{da(t)}{dt} = \frac{\mathrm{i}}{\hbar}[H_{\mathrm{Driven}}, a(t)] \tag{10.5.29}$$

$$= -\mathrm{i}\omega a(t) - \mathrm{i}g(t), \tag{10.5.30}$$

which has the solution

$$a(t) = \mathrm{e}^{-\mathrm{i}\omega t}a(0) - \mathrm{i}\int_0^t dt'\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}g(t'), \tag{10.5.31}$$

$$\equiv \mathrm{e}^{-\mathrm{i}\omega t}a(0) + \alpha(t). \tag{10.5.32}$$

a) Initial Vacuum State: If the initial state of the system corresponds to $|\mathrm{vac}, 0\rangle$, the vacuum of $a(0)$, then initially

$$a(0)|\mathrm{vac}, 0\rangle = 0. \tag{10.5.33}$$

Thus, at the time t we have

$$a(t)|\mathrm{vac}, 0\rangle = \alpha(t)|\mathrm{vac}, 0\rangle. \tag{10.5.34}$$

This means that this state is equivalent to the coherent state of argument $\alpha(t)$ with respect to the the Heisenberg operator at time t, that is

$$|\mathrm{vac}, 0\rangle \equiv |\alpha(t), t\rangle. \tag{10.5.35}$$

Exercise 10.4 Schrödinger Picture: Show that it follows from the Heisenberg picture result (10.5.34), that in the Schrödinger picture the states are described in terms of coherent states of the time-independent operators $a, a^\dagger$, and the solution of the Schrödinger equation

$$\mathrm{i}\hbar\frac{d|\psi, t\rangle}{dt} = H_{\mathrm{Driven}}|\psi, t\rangle, \tag{10.5.36}$$

is the coherent state $|\alpha(t)\rangle$.

b) Initial Coherent State: If the Schrödinger picture initial state is the coherent state $|\alpha_0\rangle$, then it is straightforward to show using (10.5.32) that the solution of the Schrödinger equation is

$$|\psi, t\rangle = \left|\alpha_0 e^{-i\omega t} + \alpha(t)\right\rangle. \tag{10.5.37}$$

10.6 Fluctuations in Systems of Fermions

If we take instead Fermi particles the results are, in a formal mathematical sense, surprisingly similar to those for Bosons. Of course the physical consequences of the differences that are in fact present, are profound.

10.6.1 The Single-Mode System

a) Moments: As in (10.3.4), we find

$$\bar{n} = \frac{\lambda}{1+\lambda} = \frac{1}{e^{\beta(\hbar\omega-\mu)}+1}. \tag{10.6.1}$$

Also, for consistency, we note that

$$\langle b^\dagger b^\dagger bb\rangle = \frac{1}{1+\lambda}\lambda^2 \frac{d^2}{d\lambda^2}(1+\lambda) = 0, \tag{10.6.2}$$

a result which is obvious from the anticommutation relations.

b) Variance: Using the anticommutation relations it follows that

$$0 = \langle b^\dagger b^\dagger bb\rangle = -\langle b^\dagger bb^\dagger b\rangle + \langle b^\dagger b\rangle, \tag{10.6.3}$$

so that $\langle N^2\rangle = \bar{n}$ and thus

$$\mathrm{var}[N] = \bar{n} - \bar{n}^2, \tag{10.6.4}$$

which differs from (10.3.7), the corresponding result for Bosons, only by the sign of the term in $\bar{n}^2$.

10.6.2 Fermi–Gaussian Systems

We define a Fermi–Gaussian density operator in a similar way to that for Bosons, except that there are no Fermi mean fields. A Fermi–Gaussian density operator therefore has the form

$$\rho = \mathcal{N}\exp\left\{-\sum_{ij}\left(A_{ij}b_i^\dagger b_j + B_{ij}b_i^\dagger b_j^\dagger + B_{ij}^* b_i b_j\right)\right\}. \tag{10.6.5}$$

a) Diagonal Fermi–Gaussian Density Operator: This takes the form

$$\rho_{\mathrm{diag}} = \mathcal{N}\exp\left\{-\sum_i \kappa_i b_i^\dagger b_i\right\}, \tag{10.6.6}$$

where the eigenvalues $\kappa_i > 0$ characterize the density operator.